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The hydrodynamics of a two-dimensional hexatic phase

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Abstract. The paper presents a simplified theoretical analysis of the hydrodynamics of a two-dimensional hexatic phase, based on the more detailed theory of Zippelius *et al* (Zippelius A, Halperin B I and Nelson D R 1980 *Phys. Rev.* B **22** 2514). The crossover, with increasing frequency, of the hydrodynamic modes from those typical of a hexatic phase to those typical of a solid is studied in a free-dislocation model, and it is discussed whether in practice this crossover is likely to be strongly affected by the presence of dislocation pairs. The theory is applied to a two-dimensional sheet of ions trapped below the free surface of superfluid helium, and the effect of both a magnetic field and a finite ionic mobility is examined. The extent to which this system offers the possibility of experimental verification of the theory is discussed.

1. Introduction

There has been much interest for many years in the process of melting in two-dimensional systems (see, e.g., Strandburg 1988), and in the possibility that this melting process might lead to the production of a new type of phase, intermediate between a crystal and a fluid. More specifically, it has been suggested that melting in two dimensions may take place through two transitions, at temperatures T_m and T_i. Below the temperature T_m there would be a crystal with quasi-long-range translational order and long-range order in the orientation of the bonds between neighbouring atoms. Between T_m and T_i there would be the intermediate, or *hexatic*, phase, with no long-range translational order, but with quasi-long-range orientational order. Above T_i there would be no long-range order at all. Melting at T_m would take place through a Kosterlitz–Thouless transition at which bound pairs of dislocations become unbound; the transition at T_i would be a second Kosterlitz–Thouless transition at which free dislocations, which can be regarded as bound pairs of disclinations, dissociate into free disclinations.

This paper is ultimately concerned with a particular type of two-dimensional system: one in which the particles are charged and where the particle interactions are due simply to Coulomb repulsion. Examples of such a system are provided by electrons trapped above the free surface of liquid helium (Grimes and Adams 1976), by ions trapped below the free surface of superfluid helium (Barenghi *et al* 1991), and by two-dimensional colloidal suspensions of charged polystyrene spheres (Murray and van Winkle 1987). The formation and melting of a crystal phase has been observed in all three types of system (Grimes and Adams 1979, Vinen *et al* 1994, Murray and van Winkle 1987). There is good evidence from direct visual observations that a hexatic phase exists in the colloidal suspensions (Murray and van Winkle 1987), but evidence for a hexatic phase in the other systems has still to be found.

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The hexatic phase has been observed and studied in other types of two-dimensional system: for example, in the flux lines present in thin films of high-temperature superconductors in an applied magnetic field (Grier *et al* 1991, Theunissen *et al* 1996).

The hexatic phase ought to exhibit special hydrodynamic properties, the observation of which would be interesting and important in itself and might provide evidence for the existence of the phase in systems where direct visual observation is not possible. The general form of the hydrodynamic equations for a hexatic phase were discussed by Zippelius *et al* (1980); they discussed not only general phenomenology but also a model in which the hexatic phase is represented as a crystal containing free dislocations (see also Stoof *et al* 1996). Satisfactory observation of the hydrodynamics of the colloidal suspensions is hardly possible owing to the damping imposed by the background medium, and the interpretation of experiments on the electron system above the surface of liquid helium is complicated by strong ripplon interactions. It has been shown recently that the ion system beneath the surface of superfluid helium suffers less seriously from these problems: at low temperatures the ions have a very high mobility, the superfluid imposing only a very small drag force, and ripplon interactions are much less severe in their effects. Preliminary experiments have already been reported in which shear mode propagation has been observed at temperatures just above T_m, and further experimental work is now in progress. This paper is intended to provide a theoretical background for these experiments. We shall review the results obtained by Zippelius *et al* and discuss how they apply to the ion system. In one sense the review is a simplified version of the work of Zippelius *et al*, although we hope that it retains enough of the essential physics, while being more accessible to the general reader. On the other hand we shall discuss some issues, notably those in sections 4.3, 4.4, and 4.6, either in more detail and or in a way that will prove more useful in a later paper where we shall compare the theory with the results of the ion experiment. Sections 5 and 6 contain new material.

The hexatic order parameter is introduced in section 2, and the way in which hexatic ordering affects the phenomenology of the hydrodynamic equations is described in section 3. The free-dislocation model is then used to obtain more detailed information about the form of the hydrodynamics and about the values of the kinetic coefficients (section 4). The effects of a vertical magnetic field and a finite particle mobility are described in some detail in section 5. The possibility of performing relevant experiments on the ion system is discussed in section 6.

2. The hexatic order parameter

The ordering in the hexatic phase is characterized by an order parameter $\Psi(\boldsymbol{r})$, which is defined as follows. We deal with a triangular lattice, such as is present in the crystal phase of the ion system, so each particle has six nearest neighbours in the crystal phase. Consider a particle (ion) at the point $\boldsymbol{r}$. Let $\theta(\boldsymbol{r})$ be the angle of orientation, relative to some fixed axis, of the bond joining this particle to one of its neighbours. Now define the bond-orientation order parameter at the position $\boldsymbol{r}$ as

$$\Psi(\boldsymbol{r}) = |\Psi| \exp[6\mathrm{i}\,\Theta(\boldsymbol{r})] = \langle \exp[6\mathrm{i}\,\theta(\boldsymbol{r})] \rangle \tag{2.1}$$

where $\langle \cdots \rangle$ represents an average over the different bonds and over thermal fluctuations. The angle $\Theta(\boldsymbol{r})$ defined by (2.1) is then a measure of the average local orientation of the bonds, while $|\Psi|$ is a measure of the extent of hexatic ordering. As is the case with other order parameters, $|\Psi|$ will decrease with increasing temperature, vanishing at the temperature T_i.

The hexatic has its minimum free energy when $\Theta(\boldsymbol{r})$ is spatially uniform. Any gradients

in $\Theta(\boldsymbol{r})$ will lead to an increase in free energy, given by

$$F_H = \frac{1}{2}K_A(T)\int |\boldsymbol{\nabla}\Theta|^2 \, \mathrm{d}^2 r \tag{2.2}$$

where the integration is over the whole area of the hexatic. The parameter $K_A(T)$ decreases with increasing temperature, as the hexatic becomes less ordered.

3. Phenomenological hydrodynamic equations and wave propagation

A straightforward generalization of the equations written down by Zippelius *et al* (1980) leads to the following equations describing the linearized dynamical behaviour of a hexatic composed of charged particles such as ions:

$$\frac{\partial \rho}{\partial t} + \rho_0 \operatorname{div} \boldsymbol{v} = 0 \tag{3.1}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + \frac{1}{\tau}\boldsymbol{v} = -\frac{c_\ell^2}{\rho_0}\boldsymbol{\nabla}\rho + \omega_c \boldsymbol{v} \times \hat{\boldsymbol{z}} + \frac{\eta}{\rho_0}\nabla^2 \boldsymbol{v} + \frac{\zeta}{\rho_0}\operatorname{grad}\operatorname{div}\boldsymbol{v} + \frac{K_A}{2\rho_0}\operatorname{curl}(\hat{\boldsymbol{z}}\,\nabla^2\Theta) \tag{3.2}$$

$$\frac{\partial \Theta}{\partial t} = \frac{1}{2}\hat{\boldsymbol{z}} \cdot \operatorname{curl}\boldsymbol{v} + \kappa\,\nabla^2\Theta. \tag{3.3}$$

ρ is the areal mass density of the hexatic, ρ_0 is the equilibrium density, $\boldsymbol{v}$ is the velocity field, c_ℓ is the speed of sound in the fluid phase, $\hat{\boldsymbol{z}}$ is the unit vector normal to the plane of the hexatic, η and ζ are coefficients of first and second viscosity, and κ is a diffusion coefficient. We have included the effect of a magnetic field, B_z, pointing normal to the plane of the hexatic, because many of the relevant experiments on the ion system are carried out in the presence of such a field; ω_c is the cyclotron frequency, eB_z/m, e and m^* being the charge and mass of each particle. We have assumed for simplicity that there is a local relationship between fluctuations in the density of the system and the corresponding fluctuations in pressure or electrostatic potential that give rise to the driving force appearing as the first term on the right-hand side of equation (3.2). (For the ion system this means that equation (3.2) is strictly true only in the limit in which the wavenumber of the disturbance is small compared with the reciprocal of the spacing between the electrodes that provide the holding field for the ions; see, e.g., Barenghi *et al* (1991).) It is assumed that the particles of the system have a finite mobility; i.e. that the motion of an ion is subject to a frictional drag against a fixed background, given by the term $(1/\tau)\boldsymbol{v}$ in equation (3.2). In practice for trapped ions below the surface of superfluid helium this drag is due at the relevant low temperatures to ripplon scattering. Further discussion of equations (3.2) and (3.3), at a phenomenological level, is given in appendix A.

We shall be particularly interested in modes of wave propagation in the hexatic, with space and time dependence $\exp(\mathrm{i}qx - \mathrm{i}\omega t)$. In the special case where $\tau = \infty$, $\omega_c = 0$ there are pure longitudinal and pure transverse modes, with the dispersion relations (Zippelius *et al* 1980)

$$\omega_\ell = c_\ell q - \frac{\mathrm{i}}{2}\left(\frac{\eta + \zeta}{\rho_0}\right)q^2 \tag{3.4}$$

and

$$\omega_t = -\frac{\mathrm{i}q^2}{2}\left[\frac{\eta}{\rho_0} + \kappa \pm \left\{\left(\frac{\eta}{\rho_0} - \kappa\right)^2 - \frac{K_A}{\rho_0}\right\}^{1/2}\right]. \tag{3.5}$$

We note that there are two transverse modes, which we shall discuss later. We shall find later that these equations, and the hydrodynamic equations on which they are based, are valid only at sufficiently low frequencies.

In the next section, where we describe the free-dislocation model, we shall retain the assumptions that $\tau = \infty$, $\omega_c = 0$. The effect of relaxing these conditions is described in section 5.

4. The free-dislocation model

Following Zippelius *et al* (1980) we model the hexatic as a crystal containing a density n_f of free, non-interacting, mobile dislocations which form at the Kosterlitz–Thouless melting transition T_m. For the moment we ignore the presence of any residual bound pairs of dislocations. Let the particle displacement field u_k describe the displacements of the particles from the sites of the perfect crystal. As mentioned in appendix A, the presence of dislocations leads to a situation where the displacement field u_k no longer corresponds to a displacement of the local centre of mass and, as explained in detail by Zippelius *et al*, is no longer a single-valued function of position. However, single-valued and continuous time and space derivatives can still be defined, except at the core of each dislocation. Let B_j be the j-component of the total Burgers vector associated with the dislocations in unit area, and let J_j^i be the corresponding ith component of the Burgers current arising from the flow of dislocations; the two are connected by the continuity equation

$$\frac{\partial B_i}{\partial t} + \frac{\partial J_j^i}{\partial x_j} = 0. \tag{4.1}$$

Let $w_{ik} = \partial u_k/\partial x_i$, the local microscopic strain field being then equal to $(w_{ik} + w_{ki})/2$. If the particle displacements are due entirely to the presence of dislocations, then (Zippelius *et al* 1980)

$$\varepsilon_{ki}\frac{\partial w_{ij}}{\partial x_k} = B_j. \tag{4.2}$$

In addition, the time derivative of w_{ij} is given by

$$\frac{\partial w_{ij}}{\partial t} - \varepsilon_{ik}J_k^j = \frac{\partial v_j}{\partial x_i} \tag{4.3}$$

where v_j is a component of the velocity of the local centre of mass of the system (equivalent to the velocity $\boldsymbol{v}$ that appears in the hydrodynamic equations).

For a two-dimensional isotropic solid the stress tensor is given in terms of the strain $u_{ij} = \frac{1}{2}(w_{ij} + w_{ji})$ by

$$\sigma_{ij} = (\lambda + \mu)u_{\ell\ell}\delta_{ij} + 2\mu\left(u_{ij} - \frac{1}{2}u_{\ell\ell}\delta_{ij}\right) \tag{4.4}$$

where λ and μ are the Lamé coefficients; μ is the shear modulus, and $K = \lambda + \mu$ is the bulk modulus.

The local mass current is governed by the usual equation of motion

$$\rho_0\frac{\partial v_i}{\partial t} = \frac{\partial \sigma_{ij}}{\partial x_j}. \tag{4.5}$$

4.1. Shear modes

Let us first confine our attention to shear motion, for which we need consider only the off-diagonal components of the stress and strain tensors. It then follows from (4.3) and (4.4) that

$$\frac{1}{\mu}\frac{\partial \sigma_{ij}}{\partial t} = \frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} + \varepsilon_{ik} J_k^j + \varepsilon_{jk} J_k^i. \tag{4.6}$$

In the presence of a shear stress a dislocation with Burgers vector b_j will experience a Peach–Koehler force equal to $-\varepsilon_{ki}\sigma_{jk}b_j$ (Nabarro 1967). Furthermore, dislocations can diffuse, with diffusion coefficient D. It follows, subject to certain assumptions that we explain in section 4.4, that the Burgers current will be given by

$$J_i^j = -\alpha\varepsilon_{ki}\sigma_{jk} - D\frac{\partial B_j}{\partial x_i} \tag{4.7}$$

where α is a dislocation mobility. Equations (4.1), (4.4), (4.5) and (4.7) form a closed system of equations for determining J_k^i, σ_{ij}, v_i and the perturbation in B_j.

4.2. Shear modes at low frequencies: comparison with phenomenological hydrodynamics

We now show that for the case of shear motion this system of equations is equivalent to the hydrodynamic equations (3.3), (3.4) and (3.5) provided that frequencies are sufficiently small. We note first that perturbations in the hexatic angle are given by

$$\delta\Theta = \frac{1}{2}\varepsilon_{ik}w_{ik}. \tag{4.8}$$

For the sake of simplicity let us suppose that the perturbations depend only on $x_1 = x$. Using equation (4.3) we then find that

$$\frac{\partial\Theta}{\partial t} = \frac{1}{2}\left(\frac{\partial v_y}{\partial x} + J_x^x + J_y^y\right) \tag{4.9}$$

while equations (4.5), (4.6) and (4.7) become

$$\frac{1}{\mu}\frac{\partial\sigma_{xy}}{\partial t} = \frac{\partial v_y}{\partial x} + J_y^y - J_x^x \tag{4.10}$$

$$J_x^x = \alpha\sigma_{xy} - D\frac{\partial B_x}{\partial x} \qquad J_y^y = -\alpha\sigma_{xy} \tag{4.11}$$

$$\rho_0\frac{\partial v_y}{\partial t} = \frac{\partial\sigma_{xy}}{\partial x}. \tag{4.12}$$

Using (4.11) to eliminate the Burgers currents, we find from equations (4.1), (4.9) and (4.10)

$$\frac{\partial B_x}{\partial t} = -\alpha\frac{\partial\sigma_{xy}}{\partial x} + D\frac{\partial^2 B_x}{\partial x^2} \tag{4.13}$$

$$\frac{\partial\Theta}{\partial t} = \frac{1}{2}\left(\frac{\partial v_y}{\partial x} - D\frac{\partial B_x}{\partial x}\right) \tag{4.14}$$

and

$$\frac{1}{\mu}\frac{\partial\sigma_{xy}}{\partial t} = \frac{\partial v_y}{\partial x} - 2\alpha\sigma_{xy} + D\frac{\partial B_x}{\partial x}. \tag{4.15}$$

We shall be particularly interested in modes of oscillation of the hexatic phase, and we shall suppose that the frequency is sufficiently low that the left-hand side of equation (4.15) is negligibly small. Then the stress tensor is given by

$$\sigma_{xy} = \frac{1}{2\alpha}\left(\frac{\partial v_y}{\partial x} + D\frac{\partial B_x}{\partial x}\right). \tag{4.16}$$

From (4.16) and (4.13) we find

$$\frac{\partial^2 v_y}{\partial x^2} = -2\frac{\partial B_x}{\partial t} + D\frac{\partial^2 B_x}{\partial x^2}. \tag{4.17}$$

It follows from (4.14) and ((4.17) that

$$\frac{\partial^2 \Theta}{\partial x\,\partial t} = -\frac{\partial B_x}{\partial t} \tag{4.18a}$$

that is

$$B_x = -\frac{\partial \Theta}{\partial x} \tag{4.18b}$$

so in the low-frequency limit the gradient in the hexatic angle is simply related to the net density of dislocations.

With the help of equations (4.18) we can write equations (4.14) and (4.16) as

$$\frac{\partial \Theta}{\partial t} = \frac{1}{2}\left(\frac{\partial v_y}{\partial x} + D\frac{\partial^2 \Theta}{\partial x^2}\right) \tag{4.19}$$

$$\sigma_{xy} = \frac{1}{2\alpha}\left(\frac{\partial v_y}{\partial x} - D\frac{\partial^2 \Theta}{\partial x^2}\right). \tag{4.20}$$

We see that equations (4.19) and (4.20) are consistent with the hydrodynamic equations (3.2) and (3.3), if we take

$$K_A = \frac{D}{\alpha} \qquad \eta = \frac{1}{2\alpha} \qquad \kappa = \frac{D}{2}. \tag{4.21}$$

We emphasize that this consistency holds only at sufficiently low frequencies.

4.3. Shear modes at high frequencies: viscoelastic behaviour

In the limit of high frequencies the left-hand side of equation (4.15) must become more important than either of the second two terms on the right-hand side, and the hexatic behaves like a solid. The hexatic therefore exhibits viscoelastic behaviour.

In order to demonstrate the crossover between low-frequency and high-frequency behaviour, we consider the propagation of a plane transverse wave with x- and t-dependence as $\exp(\mathrm{i}qx - \mathrm{i}\omega t)$. Equations (4.12), (4.13) and (4.15) become

$$-\mathrm{i}\omega\rho_0 v_y = \mathrm{i}q\sigma_{xy} \tag{4.22}$$

$$-\mathrm{i}\omega B_x + Dq^2 B_x = -\mathrm{i}q\alpha\sigma_{xy} \tag{4.23}$$

$$-\frac{\mathrm{i}\omega}{\mu}\sigma_{xy} = \mathrm{i}q v_y - 2\alpha\sigma_{xy} + \mathrm{i}qDB_x. \tag{4.24}$$

Eliminating v_y, B_x and σ_{xy} we obtain the dispersion relation

$$\frac{\omega^2}{c_t^2} - q^2 + \frac{\mathrm{i}\omega}{\nu}\frac{\mathrm{i}\omega - \frac{1}{2}Dq^2}{\mathrm{i}\omega - Dq^2} = 0 \tag{4.25}$$

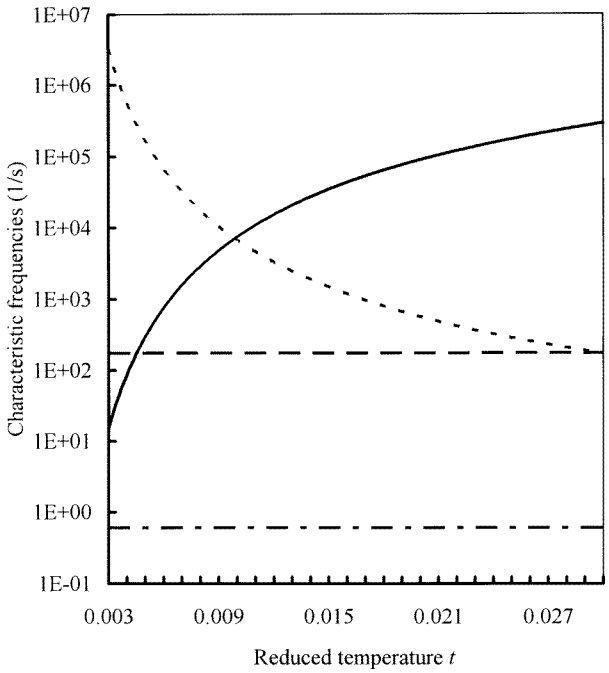

Figure 1. Characteristic frequencies plotted against reduced temperature $t = T/T_m$; $q = 10^3\ \mathrm{m}^{-1}$. Full curve: ω_s; long-dashed curve: $1/\tau$; short-dashed curve: ω_1; chain curve: ω_2.

where $c_t = (\mu/\rho_0)^{1/2}$ is the speed of transverse sound in a solid with shear modulus μ, and $\nu = \eta/\rho_0$ is the kinematic viscosity of a fluid with viscosity η. In practice (see figure 1) Dq^2 is likely to be much smaller than ω. We see then that crossover between viscous ($\omega = -\mathrm{i}\nu q^2$) and elastic ($\omega = c_t q$) behaviour takes place when $q \approx q_s = c_t/\nu$ or $\omega \approx \omega_s = c_t^2/\nu$. In the low-frequency limit $\omega \ll \omega_s$, equation (4.25) reduces to (3.5), provided that we use the relationships (4.21). We emphasize that the hydrodynamic equations of section 3 hold only in the low-frequency limit.

4.4. Longitudinal modes

A discussion of longitudinal modes is straightforward except insofar as care is required in writing down the analogue of equation (4.7), which describes the motion of dislocations induced by diffusion and by a strain in the lattice.

We first remark that equation (4.7) is itself an oversimplification because it incorporates the assumption that dislocations have associated with them a single mobility and a single diffusion coefficient. In reality, we ought to distinguish between the motion of a dislocation in a direction parallel to the Burgers vector (glide) from that in a direction perpendicular to the Burgers vector (climb). In the latter case motion can take place only if vacancies or interstitials are absorbed or emitted. These emission or absorption processes must impede the motion of the dislocation, so the corresponding mobility must be reduced. We ignored this fact in (4.7), making the assumption that there is only a single mobility.

In the case of shear modes the dislocations are driven by off-diagonal components of the shear stress, and we have only the Peach–Koehler force. In the case of longitudinal modes we must recognize that a dislocation can be driven in a direction perpendicular to

its Burgers vector by a gradient in the density of point defects. Here we shall not consider the general case, but instead we shall assume that the point defects are always close to thermal equilibrium. As shown by Zippelius *et al*, the induced Burgers currents can then be expressed in terms of effective Peach–Koehler forces involving the diagonal components of the stress tensor, together of course with the effect of diffusion. Equations (4.26) and (4.27) are the result, where we have again assumed that there are single values of the mobility and diffusion coefficients.

We shall take the longitudinal mode to involve dependences on x and t as given by $\exp(\mathrm{i}qx - \mathrm{i}\omega t)$. In accord with the argument that we have just mentioned, the Burgers currents are then given by

$$J_y^x = \frac{\alpha}{2}(\sigma_{yy} - \sigma_{xx}) \tag{4.26}$$

and

$$J_x^y = \frac{\alpha}{2}(\sigma_{yy} - \sigma_{xx}) - \mathrm{i}qDB_y. \tag{4.27}$$

In addition we have from equations (4.5), (4.1), (4.3) and (4.4)

$$-\mathrm{i}\omega\rho_0 v_x = \mathrm{i}q\sigma_{xx} \tag{4.28}$$

$$-\mathrm{i}\omega B_y = -\mathrm{i}qJ_x^y \tag{4.29}$$

$$-\mathrm{i}\omega\sigma_{xx} = (2\mu + \lambda)(\mathrm{i}qv_x + J_y^x) - \lambda J_x^y \tag{4.30}$$

$$-\mathrm{i}\omega\sigma_{yy} = \lambda(\mathrm{i}qv_x + J_y^x) - (2\mu + \lambda)J_x^y. \tag{4.31}$$

Elimination of v_x, B_y, J_x^y and J_y^x leads to equations for σ_{xx} and σ_{yy}, from which we obtain the following dispersion relation

$$\begin{vmatrix} -\omega^2 + \dfrac{2\mu+\lambda}{\rho_0}q^2 - \dfrac{\mathrm{i}\omega\alpha}{2}\left(2\mu - \dfrac{\lambda Dq^2}{\mathrm{i}\omega - Dq^2}\right) & \dfrac{\mathrm{i}\omega\alpha}{2}\left(2\mu - \dfrac{\lambda Dq^2}{\mathrm{i}\omega - Dq^2}\right) \\ \dfrac{\lambda q^2}{\rho_0} + \dfrac{\mathrm{i}\omega\alpha}{2}\left\{2\mu + (2\mu+\lambda)\dfrac{Dq^2}{\mathrm{i}\omega - Dq^2}\right\} & -\omega^2 - \dfrac{\mathrm{i}\omega\alpha}{2}\left\{2\mu + (2\mu+\lambda)\dfrac{Dq^2}{\mathrm{i}\omega - Dq^2}\right\} \end{vmatrix} = 0. \tag{4.32}$$

We expand to first order in D (assuming always that $Dq^2/\omega \ll 1$). We distinguish two cases: low frequencies ($\omega \ll \omega_s = c_t^2/v = 2\alpha\mu$); and high frequencies ($\omega \gg \omega_s$). In the low-frequency case we expand also to first order in $\omega/\alpha\mu$; in the high-frequency case we expand to first order in $\alpha\mu/\omega$. We find the following results.

For low frequencies:

$$\omega^2 - c_\ell^2 q^2 + \mathrm{i}\omega q^2\left(\frac{1}{2\alpha\rho_0} + \frac{D}{2}\right) = 0 \tag{4.33}$$

where

$$c_\ell^2 = \frac{\lambda + \mu}{\rho_0}. \tag{4.34}$$

For high frequencies:

$$\omega^2 - \tilde{c}_\ell^2 q^2 + \mathrm{i}\left(\frac{2\mu}{2\mu + \lambda}\right)\alpha\mu\omega = 0 \tag{4.35}$$

where

$$\tilde{c}_\ell^2 = \frac{\lambda + 2\mu}{\rho_0}. \tag{4.36}$$

The low-frequency limit corresponds to a motion in which $\sigma_{xx} \approx \sigma_{yy}$, which means that the hexatic is being compressed uniformly, as in sound propagation in a liquid. In the high-frequency limit $\sigma_{yy} = \sigma_{xx}\lambda/(\lambda + 2\mu)$, so, as in longitudinal sound propagation in a solid, compression is no longer uniform.

Comparing (4.33) with (3.4), and remembering (4.21), we see that the coefficient of second viscosity in the free-dislocation model must be given by

$$\zeta = \frac{\rho_0 D}{2}. \tag{4.37}$$

4.5. Values of the hydrodynamic coefficients

Let n_f be the number of free dislocations per unit area. Suppose that the movement of each dislocation by an interparticle spacing a_0 involves the surmounting of an energy barrier W with an attack frequency ω_0. A simple calculation then yields a diffusion coefficient D at temperature T given by

$$D = 2a_0^2\omega_0 \exp\left(-\frac{W}{k_B T}\right). \tag{4.38}$$

The mobility α can then be obtained from the Einstein relation, which in this case takes the form

$$D = \frac{\alpha k_B T}{n_f a_0^2}. \tag{4.39}$$

Hence we have

$$\alpha = \frac{2a_0^4 n_f \omega_0}{k_B T} \exp\left(-\frac{W}{k_B T}\right). \tag{4.40}$$

Then, using equations (4.21), we find

$$K_A = \frac{k_B T}{n_f a_0^2} \tag{4.41}$$

$$\eta = \frac{k_B T}{4a_0^4 n_f \omega_0} \exp\left(\frac{W}{k_B T}\right) \tag{4.42}$$

$$\kappa = \frac{\zeta}{\rho_0} = a_0^2\omega_0 \exp\left(-\frac{W}{k_B T}\right). \tag{4.43}$$

4.6. The effect of bound pairs of dislocations

So far we have ignored the fact that, especially near the melting temperature T_m, there is present a significant density of bound pairs of dislocations. The effect of these bound pairs was discussed briefly by Zippelius *et al* (1980) and has been discussed in more detail by Dahm *et al* (1989). Here we confine ourselves to a brief introductory discussion and to some analysis of the effect of the bound pairs on our earlier results. Our discussion is deliberately qualitative and elementary, to emphasize the essential physics; a more thorough discussion would require an approach similar to that of Ambegaokar *et al* (1980) developed in the context of the Kosterlitz–Thouless transition in superfluid films.

When the crystalline or hexatic phase is subjected to a shear stress the members of each dislocation pair will experience the corresponding Peach–Koehler force. As a result they will move until this force is balanced by the modified force of interaction between the two dislocations. The net effect is a reorientation of the dislocation pair. In the crystalline phase

and for a time-independent applied stress, the effective shear modulus is thereby reduced. For a time-dependent stress the fact that the reorientation of each dislocation takes place at a finite rate, characterized by a finite relaxation time τ_P, leads to dissipation. In the hexatic phase analogous effects take place and add to the effect of the free dislocations; in particular, there will be added dissipation in a shear mode, which will be observed as a frequency-dependent addition to the viscosity, and a modification to the effective shear modulus that enters into the crossover frequency ω_s.

In order to judge the importance of these effects in the context of our earlier discussion we need first to estimate the relaxation time τ_P. The reorientation of a dislocation pair in the external stress field will take place as a result of diffusion, which is characterized by the diffusion coefficient D. It follows that for a dislocation pair with separation d the relaxation time, $\tau_P(d)$, must be of order d^2/D. Following the work of Ambegaokar *et al* (1980) on the analogous problem relating to vortex pairs in a superfluid film, Dahm *et al* (1989) suggest that

$$\tau_P(d) = \frac{d^2}{14D}. \tag{4.44}$$

In practice dislocation pairs will be present with a range of separations, and we shall denote by $d_{\max}$ (a function of temperature) the maximum separation for which there is a significant concentration of these pairs. For the moment we shall assume that the behaviour of the dislocation pairs is dominated by those with the largest separation, and that there is therefore a single dominant relaxation time $\tau_P(d_{\max})$.

To proceed further let us imagine that we have removed all of the free dislocations from the hexatic phase, leaving the bound dislocation pairs unchanged. We introduce two values of the shear modulus of the system as it then exists: the value μ_0 that applies in the limit of zero frequency; and the value μ_∞ that applies in the limit of frequencies much larger than $1/\tau_P(d_{\max})$. The value of μ_∞ is not affected by the bound dislocation pairs: the pairs cannot follow the fast variation of the applied stress and therefore do not contribute to the linear response. In contrast, in the zero-frequency limit the bound pairs adiabatically follow the applied stress, which leads to a reduction in the shear modulus. At temperatures below T_m, $d_{\max}$ increases with increasing temperature as more and more dislocation pairs are thermally excited, the rate of increase being strongly influenced by the fact that the free energy of a dislocation pair with separation d is determined by a value of the effective shear modulus that is influenced (reduced) by the thermally excited pairs with separations less than d. This effect is similar to the reduction of the superfluid density by bound pairs of vortices (Ambegaokar *et al* 1980). The increase in density of dislocation pairs with temperature below T_m causes μ_0 to fall increasingly below μ_∞. At temperatures above T_m the density of dislocation pairs decreases, as they dissociate into free dislocations, so μ_0 returns gradually to the value μ_∞. Above T_m the value of $d_{\max}$ must be a little smaller than the spacing between the free vortices, so it must then be related to the density of free dislocations by the equation

$$d_{\max} = g n_f^{-1/2} \tag{4.45}$$

where the number g is a little less than unity. With increasing temperature within the hexatic phase, n_f must increase (see section 6.1), so $d_{\max}$ must fall.

The details underlying this picture are complicated and must be elucidated with the help of renormalization group theory, but for frequencies either much smaller than, or much larger than, $\omega_P = 1/\tau_P(d_{\max})$ the effect of the bound dislocation pairs can be incorporated into our theory simply by using the moduli $\mu = \mu_0$ or $\mu = \mu_\infty$, respectively. If the frequency is

comparable with ω_P one must use a frequency-dependent complex shear modulus $\mu(\omega)$, the imaginary part of which determines the energy dissipation. This dissipation is a maximum when $\omega \sim \omega_P$. But, if the shear modulus is frequency dependent, then the system of differential equations on which our analysis has been based becomes invalid and must be revised. Therefore our approach gives a reliable account of the crossover between viscous and elastic behaviour (sections 4.3 and 4.4) only if the frequency ω_s at which it occurs is either much larger than or much smaller than ω_P.

We show later that in the case of the ion system the values of μ_0 and μ_∞ differ by less than 40%; in calculating an approximately correct value of ω_s we shall use a shear modulus equal to μ_0. Then putting $\omega_s = \mu_0/\eta$, and using equations (4.21), (4.39), (4.44) and (4.45), we find that

$$\frac{\omega_s}{\omega_P} = \frac{g^2\mu_0 a_0^2}{7k_BT}. \tag{4.46}$$

At temperatures above, but close to T_m, where the number of free dislocations is small, we can use the critical value of μ_0, which is connected with T_m by a relation similar to that for the critical superfluid density in the Kosterlitz–Thouless theory (see Nelson and Halperin 1979):

$$\frac{\mu_0 a_0^2}{k_BT_m} = 4\pi \tag{4.47}$$

and hence

$$\frac{\omega_s}{\omega_P} = \frac{4\pi g^2}{7}. \tag{4.48}$$

This analysis is based on the assumption that there is a single relaxation time equal to $\tau_P(d_{\max})$. In reality there are a range of such times, corresponding to a range of values of d. According to Dahm and co-workers (Dahm *et al* 1989, Dahm 1997), the effect is to place the maximum dissipation at a frequency equal to approximately $25\omega_P$, so ratio (4.48) ought to be reduced by a factor of 25. This means that ω_s can be expected to be considerably less than ω_P, so the effects of relaxation of the dislocation pairs ought indeed to be distinguishable from those of the viscoelastic crossover.

5. The effect of a magnetic field and a finite particle mobility

In the phenomenological hydrodynamic equation (3.2) we included the effect of a steady magnetic field applied normal to the sheet of particles and of a finite particle mobility. We ignored these effects in our subsequent discussion, although in many experiments they are actually present. In this section, by way of illustration, we consider their effect on the propagation of transverse waves in the low-frequency limit.

We take the space and time dependence to be of the form $\exp(\mathrm{i}qx - \mathrm{i}\omega t)$. We decompose the velocity v into longitudinal and transverse components v_ℓ and v_t. Equations (3.1) to (3.2) can then be written, after elimination of ρ and Θ, as

$$\left[\omega^2 - c_\ell^2 q^2 + \frac{\mathrm{i}\omega}{\tau} + \frac{\mathrm{i}\omega}{\rho_0}(\eta+\zeta)q^2\right] v_\ell - \mathrm{i}\omega\omega_c v_t = 0 \tag{5.1}$$

$$\left[\omega^2 + \frac{\mathrm{i}\omega}{\tau} + \frac{\mathrm{i}\omega\eta q^2}{\rho_0}\right] v_t + \mathrm{i}\omega\omega_c v_\ell + \frac{\mathrm{i}\omega q^4 K_A}{4\rho_0(-\mathrm{i}\omega + \kappa q^2)} v_t = 0. \tag{5.2}$$

We shall assume that the frequency of the transverse modes of interest to us is much less than $c_\ell q$, which accords with practical situations discussed in section 6. Then to a good

approximation (5.1) reduces to

$$v_\ell = -\frac{\mathrm{i}\omega\omega_c}{c_\ell^2 q^2} v_t \tag{5.3}$$

and we find from (5.2) after some manipulation the dispersion relation

$$\omega^2 + \mathrm{i}\omega\left(\kappa q^2 + \frac{\gamma\eta q^2}{\rho_0} + \frac{\gamma}{\tau}\right) - \frac{\gamma\eta\kappa q^4}{\rho_0} - \frac{\gamma K_A q^4}{4\rho_0} = 0 \tag{5.4}$$

where $\gamma = c_\ell^2 q^2/(c_\ell^2 q^2 + \omega_c^2)$. It follows that

$$\omega = -\frac{\mathrm{i}q^2}{2}\left[\frac{\gamma\eta}{\rho_0} + \kappa + \frac{\gamma}{q^2\tau} \pm \left\{\left(\kappa + \frac{\gamma\eta}{\rho_0} + \frac{\gamma}{q^2\tau}\right)^2 - \frac{4\gamma\eta\kappa}{\rho_0} - \frac{\gamma K_A}{\rho_0}\right\}^{1/2}\right] \tag{5.5}$$

which is the required generalization of (3.5). We see that the effect of the magnetic field represented by the factor γ is different for κ and for the other hydrodynamic coefficients. This demonstrates an unsatisfactory feature of a suggestion by Stoof *et al* (1996) that the dissipation term proportional to κ can be eliminated by appropriately redefining K_A and η. Although this elimination is formally possible, it makes the theory less transparent, since the dissipation in equation (3.3) for the hexatic angle has a clear physical significance.

In practice γ is of the order of or less than unity, and we see that the magnetic field does not then have any major qualitative effect.

6. Application to the Coulomb system

In this last section we shall apply our results explicitly to a Coulomb system as exemplified by the ion system described in section 1.

We first explain that the interaction of the ions in this system with the superfluid helium in which they are trapped has only two effects that are relevant to our present discussion (Barenghi *et al* 1991): it serves to increase the effective mass of each ion; and it introduces some drag on the horizontal motion of the ions, due to the scattering of the excitations in the liquid, including the ripplons on its surface. The theory summarized in earlier sections of this paper ought therefore to describe the behaviour of any hexatic phase of the ion system.

For a typical sheet of positive ions (each with effective mass, m^*, equal to 2.15×10^{-25} kg) the ion number density, n_0, is 2.77×10^{11} m^{-2}, so $\rho_0 = 5.96 \times 10^{-14}$ kg m^{-2}. The predicted melting temperature T_m is 120 mK. The speed of the longitudinal mode (in zero magnetic field) is given by (Barenghi *et al* 1991)

$$c_\ell^2 = \frac{n_0 e^2 d}{2\varepsilon_0 m^*} \tag{6.1}$$

where the sheet is held midway between two electrodes separated by distance $2d$, and we have assumed that $qd \ll 1$. In the absence of renormalization due to dislocation pairs, the shear modulus in the crystal phase is given by

$$\mu = \mu_\infty = \frac{A(T) e^2 n_0^{3/2}}{4\pi\varepsilon_0} \tag{6.2}$$

where $A(0) = 0.245$ (Bonsall and Maradudin 1977). Phonon–phonon interactions cause $A(T)$ to fall with increasing temperature, and we shall take it to be equal to 0.18 near T_m (Morf 1979, Deville *et al* 1984). As explained in section 4.6, the presence of dislocation pairs will further reduce the shear modulus at low frequencies, the value of μ_0 at the

melting temperature T_m being given by (4.47). For the typical ion sheet, we find: $\mu_\infty(0) = 8.22 \times 10^{-12}$ N m^{-1}; $\mu_\infty(T_m) = 6.03 \times 10^{-12}$ N m^{-1}; and $\mu_0(T_m) = 4.34 \times 10^{-12}$ N m^{-1}.

The attack frequency ω_0 is probably equal to roughly the plasma mode frequency (or the shear mode frequency) in the crystal phase at the Brillouin zone boundary, and is therefore given roughly by

$$\omega_0^2 = \frac{n_0 e^2}{\varepsilon_0 m^* a_0} \tag{6.3}$$

where n_0 and a_0 are related, for a triangular lattice, by $n_0 = 4/3a_0^2$.

It is convenient to introduce the plasma parameter

$$\Gamma = \frac{n_0^{1/2} e^2}{4\pi^{1/2} \varepsilon_0 k_B T} \tag{6.4}$$

which takes the value 130 at $T = T_m$. The kinematic viscosity is then predicted to be given by

$$\nu = \frac{e}{16(\pi \varepsilon_0 m^*)^{1/2} \Gamma} \frac{n_0}{n_f} n_0^{-1/4} \exp\left(\frac{W}{k_B T}\right) \tag{6.5}$$

where we have now used the approximate relationship $n_0 = a_0^{-2}$.

The parameters κ and K_A are given by

$$\kappa = \frac{e n_0^{-1/4}}{\varepsilon_0^{1/2} m^{*1/2}} \exp\left(-\frac{W}{k_B T}\right) \tag{6.6}$$

and

$$K_A = \frac{k_B T n_0}{n_f}. \tag{6.7}$$

In order to proceed further we need to have information about the concentration n_f of free dislocations and the activation energy W.

6.1. The concentration of free dislocations and the activation energy W

As we have seen, there is no long-range translational order in the hexatic phase, owing to the presence of free dislocations. The translational order falls off as $\exp(-r/\xi_+(T))$ (Nelson and Halperin 1979). The correlation length $\xi_+(T)$ is related to the concentration of free dislocations through the equation $n_f \approx \xi_+^{-2}$. For temperatures just above T_m, to which we shall confine our attention, $\xi_+(T)$ is expected to depend on reduced temperature $t = (T - T_m)/T_m$ according to the equation

$$\xi_+(T) = \xi_0 \exp(bt^{-0.37}) \tag{6.8}$$

where ξ_0 and b are constants. The length ξ_0 is presumably of order a_0, and we shall take these quantities to be equal. The factor b is determined by the ratio $E_c/k_B T_m$, where E_c is the energy of the core of a dislocation (see, e.g., Dahm 1984). The value of E_c for a Coulomb crystal at zero temperature has been calculated by Fisher *et al* (1979). For our typical sheet of ions, for which $T_m = 161$ mK, the calculation leads to $E_c/k_B = 1.3$ K. E_c must almost certainly decrease with increasing temperature. However, the precise way in which it does so seems not to be known. This leads to serious uncertainty in the value of b and to serious and very large uncertainty in the values of $\xi_+(T)$ and n_f. Morf (1979) has suggested that E_c has fallen at T_m by a factor equal to $\mu_\infty(T_m)/\mu_\infty(0)$. This would lead to a value of b equal to about 4, which would lead in turn to values of $\xi_+(T)$ that

are larger than the size of a typical ion sheet on which experiments can be carried out over an appreciable temperature range above T_m. (At temperatures where $\xi_+(T)$ is less than the size of the sheet, t is then too large for (6.8) to be valid. Indeed the idea of a Kosterlitz–Thouless transition may then become problematical.) Whether this is really the case is not known, and further experiments are clearly required. For the purposes of illustration we shall assume, not perhaps unreasonably, that E_c has been reduced by a factor equal to $\mu_0(T_m)/\mu_\infty(0)$, which is 0.53. In that case, $E_c/k_B = 0.69$ K at $T = T_m$, which leads to a value of b that is close to unity and to much smaller values of $\xi_+(T)$, the value of $\xi_+(T)$ not exceeding the size of the ion sheet unless $t < 1.6 \times 10^{-3}$. Therefore we take

$$\frac{n_f}{n_0} = \exp(-2t^{-0.37}). \tag{6.9}$$

This relationship holds only if t is sufficiently small: according to Dahm (1984) this means $t < 0.03$.

We know of no calculations of the activation energy W. It must presumably be significantly less than E_c (Dahm *et al* 1989), and for the purposes of illustration we shall take it to be 200 mK.

6.2. Predictions for a hexatic phase of the ion sheets

We can now calculate numerical values for the parameters $\nu = \eta/\rho_0$, ζ, K_A, κ and ω_s for the ionic hexatic phase. We choose as an example our typical positive-ion sheet at the reduced temperature $t = 10^{-2}$. We find

$$\frac{n_f}{n_0} = 1.69 \times 10^{-5} \tag{6.10}$$

$$\nu = 1.36 \times 10^{-2}\ \mathrm{m^2\ s^{-1}} \tag{6.11}$$

$$\frac{\zeta}{\rho_0} = \kappa = 3.02 \times 10^{-5}\ \mathrm{m^2\ s^{-1}} \tag{6.12}$$

$$\frac{K_A}{\rho_0} = 1.64 \times 10^{-6}\ \mathrm{m^4\ s^{-1}} \tag{6.13}$$

$$\omega_s = 7.44 \times 10^3\ \mathrm{s^{-1}}. \tag{6.14}$$

We see that $\kappa \ll \nu$ and $K_A/\rho_0 \ll \nu^2$. Under these conditions the two frequencies given by (3.5) reduce to

$$\omega_{t1} = -\frac{1}{2}\mathrm{i}q^2\nu \tag{6.15}$$

and

$$\omega_{t2} = -2\mathrm{i}q^2\kappa. \tag{6.16}$$

In practical experiments one might consider trying to observe the modes (6.15) and (6.16), to obtain values of ν and κ as functions of temperature, and to observe the viscoelastic crossover frequency ω_s. In practice the wavenumber q might be determined by the size of a typical ion sheet and might therefore be equal to, say, $10^3\ \mathrm{m^{-1}}$. At any finite temperature the ions have a finite mobility, determined at the relevant temperatures by ripplon scattering. The corresponding relaxation time τ depends on trapping depth but is typically about 6×10^{-3} s at a temperature near the melting temperature (120 mK) of our typical sheet. The effect of this finite mobility can be seen from (5.5). In order that a mode frequency be not totally dominated by the finite mobility we require $\omega_t\tau \geqslant 1$. We see that the practicability of experiments can be judged by looking at the values of four characteristic frequencies:

$\omega_1 = \nu q^2/2$; $\omega_2 = 2\kappa q^2$; ω_s; and $1/\tau$. The values of these frequencies (for the case where $q = 10^3$ m^{-1}) are plotted over the reduced temperature range up to 3×10^{-2} in figure 1. We see that observation of both the mode (6.15) and the crossover frequency ω_s ought to be possible, although observation of the mode (6.16) is probably not possible. However, we emphasize that these tentative conclusions are based on a very uncertain estimate of the parameter b that determines the concentration of dislocations in the hexatic phase. Further experimental and theoretical work is clearly required.

The way in which shear modes can be excited and detected in the ion sheets has been described by Elliott *et al* (1996), who were able to verify that lightly damped shear modes, with a shear modulus given by (6.2), are indeed present in the crystal phase. Experiments are now in progress to detect the mode (6.15) and hence to determine values of both the kinematic viscosity at temperatures above T_m and the crossover frequency.

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Appendix A

We comment on the form of the hydrodynamical equations (3.2) and (3.3)

Suppose first that we are dealing with a solid, and that the increase in free energy due to bending of the interparticle bonds is given by equation (2.2). Let the bending be due to a particle displacement field $\delta \boldsymbol{u}$. We derive the resulting force acting on unit volume of the solid, as follows. The change in the angle Θ is related in an obvious way to the curl of this field; i.e.

$$\delta\Theta = \frac{1}{2}\varepsilon_{ik}\frac{\partial \delta u_k}{\partial x_i} \tag{A.1}$$

and therefore the corresponding change in the free energy F_H is given by

$$\delta F_H = \frac{1}{2}K_A \int \frac{\partial\Theta}{\partial x_j}\frac{\partial}{\partial x_j}\left(\varepsilon_{ik}\frac{\partial \delta u_k}{\partial x_i}\right) \mathrm{d}^2 r = \frac{1}{2}K_A \int \varepsilon_{ik}\frac{\partial^3\Theta}{\partial x_i\, \partial x_j^2}\,\delta u_k \,\mathrm{d}^2 r \tag{A.2}$$

where we have made use of elementary vector transformations and neglected the effect of any forces at the edge of the hexatic. We can rewrite (A.2) in the form

$$\delta F_H = -\int \frac{\partial \sigma_{kj}}{\partial x_j}\,\delta u_k \,\mathrm{d}^2 r \tag{A.3}$$

where the stress tensor, σ_{jk}, is given by

$$\sigma_{kj} = \frac{1}{2}K_A \varepsilon_{ki}\frac{\partial^2\Theta}{\partial x_i\, \partial x_j}. \tag{A.4}$$

It follows that the force per unit area acting on the hexatic must be given by

$$f_k = -\frac{\partial \sigma_{kj}}{\partial x_j} \tag{A.5}$$

which is equivalent to a force per unit mass equal to the last term in equation (3.4). However, in order to satisfy conservation of angular momentum, the stress tensor (A.4) ought to be

symmetric (Landau and Lifshitz 1970), and this can be achieved without altering f_k by adding a second term, so

$$\sigma_{kj} = \frac{1}{2}K_A\left(\varepsilon_{ki}\frac{\partial^2\Theta}{\partial x_i\,\partial x_j} + \varepsilon_{ji}\frac{\partial^2\Theta}{\partial x_i\,\partial x_k}\right). \tag{A.6}$$

This is the form of the stress tensor used by Zippelius *et al* (1980).

In the solid, changes in the angle Θ are directly related to the rotation of the solid through equation (A.1) from which it follows that

$$\frac{\partial\Theta}{\partial t} = \frac{1}{2}\hat{\boldsymbol{z}}\cdot\operatorname{curl}\boldsymbol{v}. \tag{A.7}$$

This relationship does not hold in the hexatic phase, as we see from our analysis of the free-dislocation model in section 4. The angle Θ is then connected to the independent degree of freedom associated with the hexatic order parameter, and we can think of rotation of the hexatic degree of freedom as giving rise to a frictional couple per unit volume equal to

$$G\left(\frac{\partial\Theta}{\partial t} - \frac{1}{2}\hat{\boldsymbol{z}}\cdot\operatorname{curl}\boldsymbol{v}\right)$$

acting on the centre-of-mass motion of the system, where G is a constant. The equation of motion then becomes

$$\frac{\partial\boldsymbol{v}}{\partial t} + \frac{\boldsymbol{v}}{\tau} = -\frac{c^2}{\rho_0}\boldsymbol{\nabla}\rho + \omega_c\boldsymbol{v}\times\hat{\boldsymbol{z}} + \frac{\eta}{\rho_0}\nabla^2\boldsymbol{v} + \frac{\zeta}{\rho_0}\operatorname{grad}\operatorname{div}\boldsymbol{v} + \frac{1}{2}G\hat{\boldsymbol{z}}\times\boldsymbol{\nabla}\left(\frac{\partial\Theta}{\partial t} - \frac{1}{2}\hat{\boldsymbol{z}}\cdot\operatorname{curl}\boldsymbol{v}\right). \tag{A.8}$$

We can also think of the couple

$$-G\left(\frac{\partial\Theta}{\partial t} - \frac{1}{2}\hat{\boldsymbol{z}}\cdot\operatorname{curl}\boldsymbol{v}\right)$$

as being balanced by a torque $-\delta F_H/\delta\Theta$ due to the hexatic energy (2.2), and hence

$$G\left(\frac{\partial\Theta}{\partial t} - \frac{1}{2}\hat{\boldsymbol{z}}\cdot\operatorname{curl}\boldsymbol{v}\right) = K_A\,\nabla^2\Theta. \tag{A.9}$$

Equations (A.8) and (A.9) are identical to (3.2) and (3.3) if $\kappa = K_A/G$.

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